Johns Hopkins Math Tournament 2021

Individual Round: General

April 3rd, 2021

Instructions

- **Remember you must be proctored while taking the exam.**
- This test contains 12 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No outside help is allowed. This includes people, the internet, translators, books, notes, calculators, or any other computational aid. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor.
- Good luck!

1. Walter owns 11 dumbbells, which have weight 1 pound, 2 pound, ..., 10 pounds, 11 pounds. Walter wants to split his dumbbells into three groups of equal total weight. What is the smallest possible product that the dumbbell weights in any one of these groups can have?

2. P and Q are the midpoints of sides AB and BC respectively in a triangle $\triangle ABC$. Suppose $\angle A = 30^\circ$, and $\angle PQC = 110^\circ$. Find $\angle B$ in degrees.

3. Keith decides that a sequence of digits is "slick" if every pair of adjacent digits in the sequence is divisible by either 23 or 17. What is the greatest possible number of "2" digits in a 2021-digit long "slick" sequence?

4. Let (a_n) be a sequence of numbers such that $a_{n+2} = 2a_n$ for all integers n. Suppose $a_1 = 1$, and $a_2 = 3$, then let $\sum_{n=1}^{2021} a_{2n} = c$, and $\sum_{n=1}^{2021} a_{2n-1} = b$. If the expression $c - b + \frac{c-b}{b}$ can be expressed as x^y for integers x and y such that x is as small as possible, what is $x + y$?

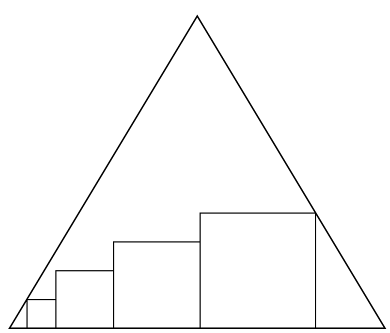
5. Terry decides to practice his arithmetic by adding the numbers between 10 and 99 inclusive. However, he accidentally swaps the digits of one of the numbers, and thus gets the incorrect sum of 4941. What is the largest possible number whose digits Terry could have swapped in the summation?

6. Alice and Bob are put in charge of building a bridge with their respective teams. With both team's combined effort, the team can be finished in 6 days. In reality, Alice's team works alone for the first 3 days, then decides to take a break. Bob's team takes over from there, and works for another 4 days. After that, they've successfully constructed 60% of the bridge. How many days would it take for Alice's team to finish building the bridge completely from the start, if Bob's team was never involved?

7. There are 4 boys and 3 girls. Each boy picks a girl, and each girl picks a boy. Assuming that each choice is uniformly random, the probability that at least one boy and one girl choose each other can be written as $\frac{p}{q}$ for relatively prime p and q. Compute $p + q$.

8. Sasha has a bag that holds 6 red marbles and 7 green marbles. How many ways can Sasha pick a handful of (zero or more) marbles from the bag such that her handful contains at least as many red marbles as green marbles? (Note: any two marbles are distinguishable, even if they have the same color.)

9. In the figure below, the triangle is equilateral and the 4 squares have side length 1, 2, 3 and 4. The area of the triangle can be expressed in simplest radical form as $\frac{a+b\sqrt{3}}{c}$ for integers a, b, and c. What is $a + b + c$?



10. The pDernaJJ Pharmaceutical Company produces a COVID-19 test that has a 95% accuracy rate on individuals who actually have an infection, and a 90% accuracy rate on individuals who do not have an infection. They use their test on a population of mathletes, of which 2% actually have an infection. If a test concludes that a mathlete has a COVID-19 infection, then the probability that the mathlete actually does have an infection can be expressed as a fraction in simplest terms as $\frac{a}{b}$. What is $a + b$?

11. Carter and Vivian decide to spend their afternoon listing pairs of real numbers, (a, b). Carter wants to find all (a, b) such that (a, b) lie within a circle of radius 6 centered at $(6, 6)$. Vivian hates circles, and would rather find all (a, b) such that a, b, and 6 can be the side lengths of a triangle. If Carter randomly chooses an (a, b) that satisfies his conditions, then the probability that the pair also satisfies Vivian's conditions can be expressed as $\frac{p}{q} + \frac{r}{s\pi}$, where p, q, r, and s are integers and the fractions $\frac{p}{q}$ and $\frac{r}{s}$ are expressed in simplest form. What is $p + q + r + s$?

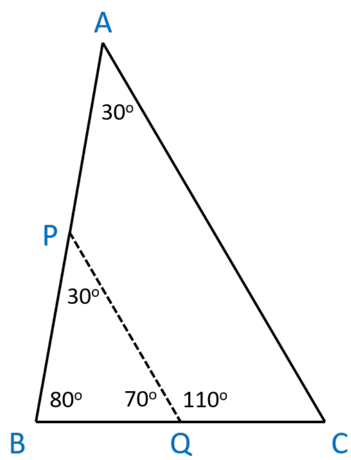
12. Let $ABCD$ be a rectangle with diagonals of length 10. Let P be the midpoint of $\overline{AD}$, let S be the midpoint of $\overline{BC}$, and let T be the midpoint of $\overline{CD}$. Points Q and R are chosen on $\overline{AB}$ such that $AP = AQ$ and $BR = BS$, and minor arcs $\widehat{PQ}$ and $\widehat{RS}$ centered at A and B, respectively, are drawn. Circle ω is tangent to $\overline{CD}$ at T, and externally tangent to $\widehat{PQ}$ and $\widehat{RS}$. Suppose that the radius of ω is $\frac{43}{18}$. Then the sum of all possible values of the area of $ABCD$ can be written in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, b and d are relatively prime, and c is prime. Find $a + b + c + d$.

General Test Solutions

1. $\boxed{110}$. First, notice that the sum of the integers $1+2+...+11 = 66$, thus each of the three groups will have a total weight of 22. In order to minimize the product of the weights in a single group, we want to set one dumbbell in the group to the lowest possible weight, or 1. Thus, the other two dumbbells will have weight 10 and 11, for a product of $1 \times 10 \times 11 = 110$.

   Note that the intuition behind setting one dumbbell to weight 1 is similar to problems in geometry in which we want to maximize the area of a shape given a set perimeter. Remember that in such geometry problems, maximizing the area happens when we set all side lengths equal (or as equal as possible given the constraints of the problem). In this problem, we are doing the opposite, we are minimize the product (i.e. the area) by making the side lengths as unequal as possible.

2. $\boxed{80}$. Refer to the diagram below for clarity. Since $\angle PQC = 110^\circ$, we know that $\angle PQB = 180^\circ - 110^\circ = 70^\circ$. Next, since P and Q are the midpoints of $\overline{AB}$ and $\overline{BC}$, we know that $\overline{PQ}$ is parallel to $\overline{AC}$. Thus, since $\angle A = 30^\circ$, it is also true that $\angle BPQ = 30^\circ$. Thus, in triangle $\triangle BPQ$, we have one angle of measure 30°, and one angle of measure 70°. Thus, the last angle, $\angle B$, must have measure $180^\circ - 30^\circ - 70^\circ = 80^\circ$.



3. $\boxed{405}$. First, since we are considering only adjacent digits in the sequence, we only care about two-digit multiples of 23 and 17. Thus, we will start by listing all the two-digit multiples of these two numbers (note that the multiples of 23 are in the left column, and multiples of 17 are in the right column):

   - 23
   - 46
   - 69
   - 92
   - 17

   - 34
   - 51
   - 68
   - 85

   Now, since none of these multiples have more than one 2 digit, we will start by trying to create a "slick" sequence by starting with 23. Note that the only number that can follow a 3 is 4, because as seen in the list above, 34 is the only two-digit number that is a multiple of either 23 or 17. We can continue this logic to find that if we start with 23, the first few digits of our sequence must look like:

   2346

   Now for the first time, we have two options with what digit to continue the "slick" sequence with; we can either continue with 8 or 9 since both 68 and 69 are two-digit multiples of either 23 or 17. First, consider what happens if we continue with 8:

23468517

However, from this point, since no two-digit multiple of 23 or 17 starts with a 7, we can no longer continue the sequence! Thus, let us instead continue the previous sequence with 9:

234692...

Here, we see that we loop back to 2, and this means that the digit sequence 23469 will just loop forever. Finally, recall that we originally started this sequence with 23. The other two-digit multiple that contains a 2 that we could've started with was 92, but since 92 is included anyway in the repeating sequence we created, starting with 92 will not allow us to fit more 2 digits in a specified length sequence. Thus, we conclude that the five digit sequence 23469 will repeat 404 times to create a 2020-digit long sequence, and since the 2021st digit will be a 2, we have a total of $404 + 1 = 405$ 2-digits.

4. $\boxed{2024}$. First, consider the $\sum_{n=1}^{2021} a_{2n} = c$ term. This sum is equivalent to $a_2 + a_4 + ... + a_{4042}$. Since $a_2 = 3$ and $a_{n+2} = 2a_n$, this sum can also be written as:

$$3 \times 2^0 + 3 \times 2^1 + ... + 3 \times 2^{2020} = 3 \times (2^0 + 2^1 + ... + 2^{2020})$$

Recall that the sum of the first i non-zero powers of 2 is $2^{i+1} - 1$, thus the sum above can be simplified to $3 \times (2^{2021} - 1)$. Next, we will apply a similar procedure to the $\sum_{n=1}^{2021} a_{2n-1} = b$. Since $a_1 = 1$, we can rewrite this sum as:

$$1 \times 2^0 + 1 \times 2^1 + ... + 1 \times 2^{2020} = 2^{2021} - 1$$

Finally, we can evaluate the expression $c - b + \frac{c-b}{b}$ as:

$$3 \times (2^{2021} - 1) - (2^{2021} - 1) + \frac{3 \times (2^{2021} - 1) - (2^{2021} - 1)}{2^{2021} - 1}$$

$$= 2 \times (2^{2021} - 1) + \frac{2 \times (2^{2021} - 1)}{2^{2021} - 1} = 2 \times (2^{2021} - 1) + 2 = 2^{2022} - 2 + 2 = 2^{2022}$$

5. $\boxed{59}$. First, we find that the unchanged sum of the integers from 10 to 99 inclusive is $(99 - 10 + 1) \times \frac{10+99}{2} = 90 \times \frac{109}{2} = 4905$. (the number of terms in the sum multiplied by the average value of a number in the sequence). Next, suppose that the number whose digits was swapped is AB, where A denotes the first digit and B denotes the second digit. In other words, the value of this integer is $10A + B$. When the digits were swapped, this number became BA, which has value $10B + A$. The difference between the swapped digits number and the original number is thus $10B + A - 10A - B = 9B - 9A$. Since Terry's incorrect sum was 4941, we find that $9B - 9A = 4941 - 4905 = 36$, and thus $9(B - A) = 4$. Thus, AB can be any two digit number such that B is 4 greater than A. Since we are finding the largest possible number that Terry could have swapped, we conclude that $AB = 59$.

6. $\boxed{15}$. Let A and B represent the proportion of the bridge that Alice/Bob's teams can finish in a day working alone respectively. Thus, from the problem, we can setup the following two equations:

$$6A + 6B = 1$$
$$3A + 4B = \frac{3}{5}$$

We can subtract $2\times$ the second equation from the first to get $2B = \frac{1}{5}$, and thus $B = \frac{1}{10}$. Then, we find that $A = \frac{1}{15}$. Thus, since each day Alice's team can finish $\frac{1}{15}$th of the bridge working alone, they will need 15 days to completely finish the bridge.

7. $\frac{55}{72}$, so $\boxed{127}$. We will approach the problem from the perspective of the 3 girls, and we will use complimentary counting, i.e. we will instead find the probability that no girl picks a boy that also picked her. Now, consider the following cases:

   - All 3 girls pick the same boy. In this case, the boy who was picked by all 3 girls must pick a girl who also picked him, so we skip this case.
   - Two girls pick one boy and the third girl picks another boy. First, there $4 \times \binom{3}{2}$ that two of the girls can pick one of the boys, and 3 ways the remaining girl can pick one of the remaining boys. Since each of these cases has a $(\frac{1}{4})^3$ chance of occurring, the overall probability that two girls pick one boy and the third girl picks another boy is $\frac{4\times\binom{3}{2}\times 3}{4^3} = \frac{9}{16}$.
     Next, the boy who was picked twice has a $\frac{1}{3}$ chance of not picking a girl who picked him, the boy who was picked once has a $\frac{2}{3}$ chance of not picking a girl who picked him, and the two boys who were never picked will always pick a girl who didn't pick them. Thus, the overall probabilitiy of this case is $\frac{9}{16} \times \frac{1}{3} \times \frac{2}{3} = \frac{1}{8}$.
   - All three girls pick different boys. If we imagine the three girls picking sequentially, the first girl can pick any boy, the second girl has a $\frac{3}{4}$ chance of picking a boy who hasn't been picked yet, and the third girl has a $\frac{2}{4}$ chance of picking a boy who hasn't been picked yet. Thus, the probably of this occurring is $\frac{3}{4} \times \frac{2}{4} = \frac{3}{8}$.
     Next, the three boys who were picked each have a $\frac{2}{3}$ chance of picking a girl who did not pick them (and the fourth boy who was always pick a girl who did not pick him). Thus, the overall probability of this case is $\frac{3}{8} \times (\frac{2}{3})^3 = \frac{1}{9}$.

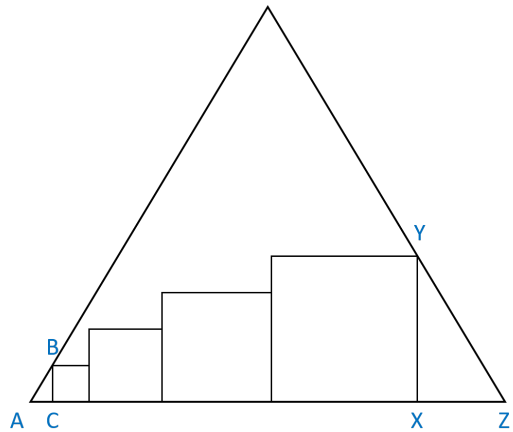
   Thus, the answer is the complement of the sum of the probabilities found in the cases above, or $1 - (\frac{1}{8} + \frac{1}{9}) = \frac{55}{72}$.

8. Suppose Sasha's handful contains r red marbles and g green marbles. Then, the bag now has $6 - r$ red marbles and $7 - g$ green marbles. If her handful has at least as many red marbles as green marbles, then $r \geq g$, so $6 - r \leq 6 - g$, so $6 - r < 7 - g$, so the bag has more green marbles than red marbles. Conversely, if her handful has more green marbles than red marbles, then $r < g$, so $6 - r > 6 - g$, so $6 - r \geq 7 - g$, so the bag has at least as many red marbles as green marbles. As such, if we let M denote the set of all 13 marbles, then for any $S \subseteq M$, exactly one of S or $M \setminus S$ has at least as many red marbles as green marbles. Thus, the number of handfuls to count is the number of distinct unordered pairs of the form $\{S, M \setminus S\}$, which is half the number of subsets of M, or $\frac{1}{2} \cdot 2^{13} = \boxed{4096}$.

9. $\boxed{637}$. Refer to the diagram below for clarity. Notice that $\triangle ABC$ and $\triangle XYZ$ are $30° - 60° - 90°$ triangles since they have a right angle and an angle of measure $60°$ (the angle shared with the equilateral triangle). Thus, since $BC = 1$, we find that $AC = \frac{\sqrt{3}}{3}$, and similarly since $XY = 4$, we find that $XZ = \frac{4\sqrt{3}}{3}$. Thus, each side of the equilateral triangle has length $1+2+3+4+\frac{\sqrt{3}}{3}+\frac{4\sqrt{3}}{3} = 10+\frac{5\sqrt{3}}{3} = \frac{30+5\sqrt{3}}{3}$. Now, since a triangle with side length s has area $s^2\frac{\sqrt{3}}{4}$, we plug this side length in and find the area as:

$$\frac{\sqrt{3}}{4}\left(\frac{30+5\sqrt{3}}{3}\right)^2 = \frac{\sqrt{3}}{4}\left(\frac{900+300\sqrt{3}+75}{9}\right)$$

$$\frac{\sqrt{3}}{4}\left(\frac{325+100\sqrt{3}}{3}\right) = \frac{325\sqrt{3}+300}{12}$$



10. $\boxed{136}$. First, we will find the probability that a test claims that an athlete has an infection. This will be the sum of the probabilities that the athlete actually does have an infection and the test was correct, and the probability that the athlete does not have an infection and the test was incorrect. Thus, we compute this probability to be:

$$\frac{1}{50} \times \frac{19}{20} + (1 - \frac{1}{50}) \times (1 - \frac{9}{10}) = \frac{117}{1000}$$

Thus, the answer will be the probability that an athlete actually does have any infection ahd the test was correct (already computed above as $\frac{1}{50} \times \frac{19}{20}$) divided by $\frac{117}{1000}$, or:

$$\frac{\frac{1}{50} \times \frac{19}{20}}{\frac{117}{1000}} = \frac{\frac{19}{1000}}{\frac{117}{1000}} = \frac{19}{117}$$

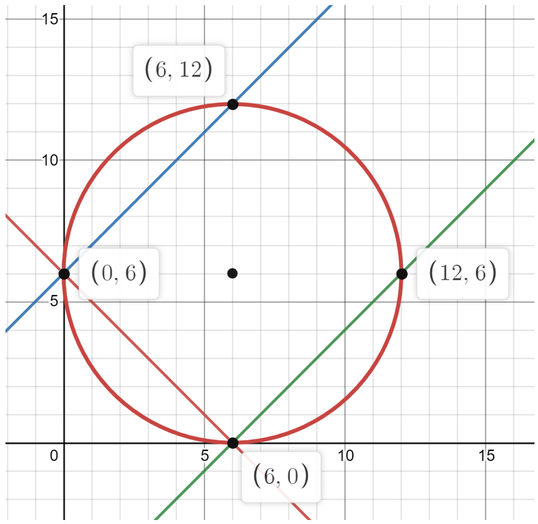
11. $\boxed{10}$. Refer to the diagram below for clarity. Consider graphing the points that satisfy Carter's conditions and the points that satisfy Vivian's conditions on the a, b-plane. Thus, Carter's points will all lie in a circle of radius 6 centered at $(6, 6)$, as specified in the problem. In order to determine the region where Vivian's points lie, we must consider the constraints given by the triangle inequality. In other words, Vivian's points must satisfy the following inequalities:

$$a + 6 \geq b$$
$$b + 6 \geq a$$
$$a + b \geq 6$$

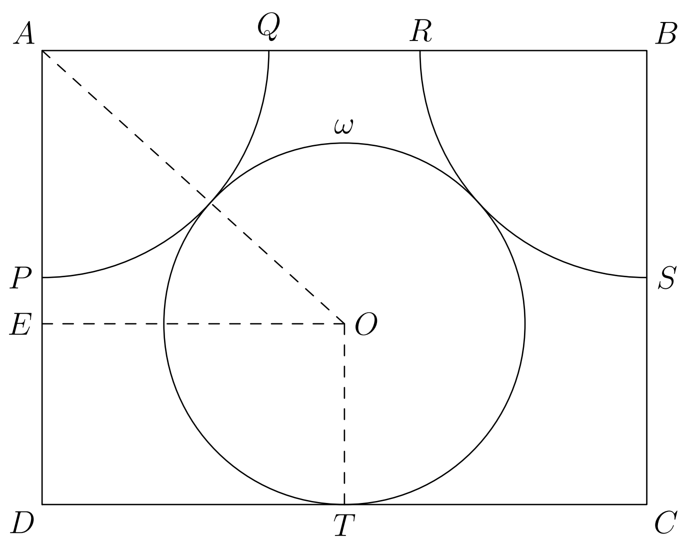
Next, to actually solve the problem, we must find the area formed by Vivian's points that are also in Carter's circle, divided by the area of Carter's circle. First, finding the area of Carter's circle is easy, it is just $\pi 6^2 = 36\pi$.

As for the area of Vivian's points inside the circle, notice that this area is the sum of a the area of a square with diagonals as diameters of Carter's circle, with a portion of the circle (bounded by the points $(6, 12)$ and $(12, 6)$ as shown in the figure). The area of this square is easy to compute - it is just $\frac{12^2}{2} = 72$. Next, the area of the portion of the circle can be found by subtracting a right triangle from a quarter of the circle, i.e. $\frac{36\pi}{4} - \frac{6\times 6}{2} = 9\pi - 18$. Thus, the area of Vivian's region is $72 + 9\pi - 18 = 54 + 9\pi$, and thus the overall probability is:

$$\frac{54 + 9\pi}{36\pi} = \frac{1}{4} + \frac{3}{2\pi}$$



12. $\boxed{577}$.



More generally, let $AB = m$, let $BC = n$, and let r be the radius of ω. Let E be a point on $\overline{AD}$ such that $\overline{EO} \perp \overline{AD}$, as shown.

Observe that $\triangle AEO$ is right. By inspection, we see that the radius of the two arcs is $\frac{n}{2}$ and $EO = \frac{m}{2}$. Then $\triangle AEO$ has leg lengths $\frac{m}{2}$ and $n - r$ and a hypotenuse of length $\frac{n}{2} + r$.

Applying the Pythagorean Theorem, we have the equation

$$\left(\frac{m}{2}\right)^2 + (n-r)^2 = \left(\frac{n}{2} + r\right)^2 \implies \frac{m^2}{4} + n^2 - 2nr + r^2 = \frac{n^2}{4} + nr + r^2 \implies 3nr = \frac{m^2 + 3n^2}{4},$$

so $r = \dfrac{m^2 + 3n^2}{12n}$.

Since $r = \dfrac{43}{18}$, we have

$$\frac{43}{18} = \frac{m^2 + 3n^2}{12n} \implies 18m^2 + 54n^2 = 516n. \tag{1}$$

Moreover, since $ABCD$ has diagonals of length 10, we have

$$m^2 + n^2 = 100. \tag{2}$$

Subtracting 18 times (2) from (1), we have

$$36n^2 = 516n - 1800 \implies 9n^2 - 129n + 450 = (n-6)(9n-75) = 0 \implies n = 6, \frac{25}{3}.$$

Thus,

$$m^2 + 6^2 = 100 \implies m^2 = 64 \implies m = 8$$

or

$$m^2 + \left(\frac{25}{3}\right)^2 = 100 \implies m^2 = \frac{275}{9} \implies m = \frac{5\sqrt{11}}{3}.$$

Therefore, the sum of all possible values of the area of $ABCD$ is

$$6(8) + \frac{25}{3}\left(\frac{5\sqrt{11}}{3}\right) = \frac{432 + 125\sqrt{11}}{9}.$$

The requested sum is $432 + 125 + 11 + 9 = 577$.